

BERGMAN KERNEL AND HYPERCONVEXITY INDEX

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Dedicated to Professor John Erik Fornaess on the occasion of his 70-th birthday

ABSTRACT. Let $\Omega \subset \mathbb{C}^n$ be a bounded domain with the hyperconvexity index $\alpha(\Omega) > 0$. Let ϱ be the relative extremal function of a fixed closed ball in Ω and set $\mu := |\varrho|(1 + |\log |\varrho||)^{-1}$, $\nu := |\varrho|(1 + |\log |\varrho||)^n$. We obtain the following estimates for the Bergman kernel: (1) For every $0 < \alpha < \alpha(\Omega)$ and $2 \leq p < 2 + \frac{2\alpha(\Omega)}{2n - \alpha(\Omega)}$, there exists a constant $C > 0$ such that $\int_{\Omega} |\frac{K_{\Omega}(\cdot, w)}{\sqrt{K_{\Omega}(w)}}|^p \leq C|\mu(w)|^{-\frac{(p-2)n}{\alpha}}$ for all $w \in \Omega$. (2) For every $0 < r < 1$, there exists a constant $C > 0$ such that $\frac{|K_{\Omega}(z, w)|^2}{K_{\Omega}(z)K_{\Omega}(w)} \leq C(\min\{\frac{\nu(z)}{\mu(w)}, \frac{\nu(w)}{\mu(z)}\})^r$ for all $z, w \in \Omega$. Various application of these estimates are given.

1. INTRODUCTION

A domain $\Omega \subset \mathbb{C}^n$ is called *hyperconvex* if there exists a negative continuous plurisubharmonic (psh) function ρ on Ω such that $\{\rho < c\} \subset\subset \Omega$ for any $c < 0$. The class of hyperconvex domains is very wide, e.g. every bounded pseudoconvex domain with Lipschitz boundary is hyperconvex (cf. [24]). Although hyperconvex domains already admit a rich function theory (see e.g. [43], [10], [33], [45]), it is not enough to get quantitative results unless one imposes certain growth conditions on the bounded exhaustion function ρ (compare [5], [9], [28]).

A meaningful condition is $-\rho \leq C\delta^{\alpha}$ for some constants $\alpha, C > 0$, where δ denotes the boundary distance. Let $\alpha(\Omega)$ be the supremum of all α . We call it the *hyperconvexity index* of Ω . From the fundamental work of Diederich-Fornaess [26], we know that if Ω is a bounded pseudoconvex domain with C^2 -boundary then there exists a continuous negative psh function ρ on Ω such that $C^{-1}\delta^{\eta} \leq -\rho \leq C\delta^{\eta}$ for some constants $\eta, C > 0$. The supremum $\eta(\Omega)$ of all η is called the *Diederich-Fornaess index* of Ω (see e.g. [1], [30], [31]). Clearly, one has $\alpha(\Omega) \geq \eta(\Omega)$. Recently, Harrington [31] showed that if Ω is a bounded pseudoconvex domain with Lipschitz boundary then $\eta(\Omega) > 0$.

On the other hand, there are plenty of domains with very irregular boundaries such that $\alpha(\Omega) > 0$, while it is difficult to verify $\eta(\Omega) > 0$. For instance, Kobe's distortion theorem implies $\alpha(\Omega) \geq 1/2$ if $\Omega \subsetneq \mathbb{C}$ is a simply-connected domain (see [15], Chapter 1, Theorem 4.4). Recently, Carleson-Totik [18] and Totik [52] obtained various Wiener-type criterions for planar domains with positive hyperconvexity indices. In particular, if $\partial\Omega$ is uniformly perfect in the sense of Pommerenke [46], then $\alpha(\Omega) > 0$ (see [18], Theorem 1.7). Moreover, for domains like $\Omega = \mathbb{C} \setminus E$, where E is a compact set in \mathbb{R} (e.g. Cantor-type sets), the connection between the metric properties of E and the precise value of $\alpha(\Omega)$ (especially the optimal case $\alpha(\Omega) = 1/2$) was studied in detail in [18] and [52]. In the appendix of this paper, we will prove $\alpha(\Omega) \geq 1/2$ for bounded \mathbb{C} -convex domains in \mathbb{C}^n and $\alpha(\Omega) > 0$

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for the total space of a holomorphic family of attracting basins at ∞ of polynomials in \mathbb{C} . Probably, the Teichmüller space of a compact Riemann surface with genus ≥ 2 has a positive hyperconvexity index.

For a domain $\Omega \subset \mathbb{C}^n$, let ϱ be the *relative extremal function* of a (fixed) closed ball $\overline{B} \subset \Omega$, i.e.,

$$\varrho(z) := \varrho_{\overline{B}}(z) := \sup\{u(z) : u \in PSH^-(\Omega), u|_{\overline{B}} \leq -1\},$$

where $PSH^-(\Omega)$ denotes the set of negative psh functions on Ω . It is known that ϱ is continuous on $\overline{\Omega}$ if Ω is a bounded hyperconvex domain (cf. [8], Proposition 3.1.3/vii)). Furthermore, it is easy to show that if $\alpha(\Omega) > 0$ then for every $0 < \alpha < \alpha(\Omega)$ there exists a constant $C > 0$ such that $-\varrho \leq C\delta^\alpha$.

The goal of this paper is to present some off-diagonal estimates of the Bergman kernel on domains with positive hyperconvexity indices, in terms of ϱ . Usually, off-diagonal behavior of the Bergman kernel is more sensitive about the geometry of a domain than on-diagonal behavior (compare [3]).

Let $K_\Omega(z, w)$ be the Bergman kernel of Ω . It is well-known that $K_\Omega(\cdot, w) \in L^2(\Omega)$ for all $w \in \Omega$. Thus it is natural to ask the following

Problem 1. *For which Ω and $p > 2$ does one have $K_\Omega(\cdot, w) \in L^p(\Omega)$ for all $w \in \Omega$?*

For the sake of convenience, we set

$$\beta(\Omega) = \sup \left\{ \beta \geq 2 : K_\Omega(\cdot, w) \in L^\beta(\Omega), \forall w \in \Omega \right\}.$$

We call it the *integrability index* of the Bergman kernel. From the well-known works of Kerzman, Catlin and Bell, we know that $\beta(\Omega) = \infty$ if Ω is a bounded pseudoconvex domain of finite D'Angelo type. On the other side, it is not difficult to see from Barrett's work [3] that there exist unbounded Diederich-Fornaess worm domains with $\beta(\Omega)$ arbitrarily close to 2 (see e.g. [37], Lemma 7.5). Thus it is meaningful to show the following

Theorem 1.1. *If $\Omega \subset \mathbb{C}^n$ is pseudoconvex, then $\beta(\Omega) \geq 2 + \frac{2\alpha(\Omega)}{2n-\alpha(\Omega)}$. Furthermore, if Ω is a bounded domain with $\alpha(\Omega) > 0$, then for every $0 < \alpha < \alpha(\Omega)$ and $2 \leq p < 2 + \frac{2\alpha(\Omega)}{2n-\alpha(\Omega)}$, there exists a constant $C > 0$ such that*

$$(1.1) \quad \int_{\Omega} |K_\Omega(\cdot, w) / \sqrt{K_\Omega(w)}|^p \leq C |\mu(w)|^{-\frac{(p-2)n}{\alpha}}, \quad w \in \Omega,$$

where $\mu := |\varrho|(1 + |\log |\varrho||)^{-1}$.

The lower bound for $\beta(\Omega)$ can be improved substantially when $n = 1$:

Theorem 1.2. *If Ω is a domain in \mathbb{C} , then $\beta(\Omega) \geq 2 + \frac{\alpha(\Omega)}{1-\alpha(\Omega)}$.*

In particular, we obtain the known fact that if $\Omega \subsetneq \mathbb{C}$ is a simply-connected domain then $\beta(\Omega) \geq 3$. A famous conjecture of Brennan [12] suggests that the bound may be improved to $\beta(\Omega) \geq 4$; an equivalent statement is that if $f : \Omega \rightarrow \mathbb{D}$ is a conformal mapping where \mathbb{D} is the unit disc, then $f' \in L^p(\Omega)$ for all $p < 4$. There is an extensive study on this conjecture (see [6], [16], [17], [47], etc.).

Nevertheless, Theorem 1.2 is best possible in view of the following

Proposition 1.3. *Let $E \subset \mathbb{C}$ be a compact set satisfying $\text{Cap}(E) > 0$ and $\dim_{\text{H}}(E) < 1$, where Cap and \dim_{H} denote the logarithmic capacity and the Hausdorff dimension respectively. Set $\Omega := \mathbb{C} \setminus E$. Then $\beta(\Omega) \leq 2 + \frac{\dim_{\text{H}}(E)}{1-\dim_{\text{H}}(E)}$.*

Example (1.1). *There exists a Cantor-type set E with $\dim_{\mathbb{H}}(E) = 0$ and $\text{Cap}(E) > 0$ (cf. [14], § 4, Theorem 5). Thus $\beta(\mathbb{C} \setminus E) = 2$ in view of Proposition 1.3.*

Example (1.2). *Andrievskii [2] constructed a compact set $E \subset \mathbb{R}$ with $\dim_{\mathbb{H}}(E) = 1/2$ and $\alpha(\mathbb{C} \setminus E) = 1/2$. It follows from Theorem 1.2 and Proposition 1.3 that $\beta(\mathbb{C} \setminus E) = 3$.*

Problem 2. *Is there a bounded domain $\Omega \subset \mathbb{C}^n$ with $\beta(\Omega) = 2$?*

The above theorems shed some light on the study of the Bergman space

$$A^p(\Omega) = \left\{ f \in \mathcal{O}(\Omega) : \int_{\Omega} |f|^p < \infty \right\}$$

for domains with positive hyperconvexity indices. For instance, we can show that $A^p(\Omega) \cap A^2(\Omega)$ lies dense in $A^2(\Omega)$ for suitable $p > 2$ and the reproducing property of $K_{\Omega}(z, w)$ holds in $A^p(\Omega)$ for suitable $p < 2$ (see § 4). Unfortunately, we don't know whether the Bergman projection can be extended to a bounded projection $L^p(\Omega) \rightarrow A^p(\Omega)$ for all p in some nonempty open interval around 2. For more information on this matter, we refer the reader to Lanzani's review article [38] and the references therein.

Set

$$K_{\Omega,p}(z) := \sup\{|f(z)| : f \in A^p(\Omega), \|f\|_{L^p(\Omega)} \leq 1\}.$$

Using $f := \frac{K_{\Omega}(\cdot, z)}{\sqrt{K_{\Omega}(z)}} / \left\| \frac{K_{\Omega}(\cdot, z)}{\sqrt{K_{\Omega}(z)}} \right\|_{L^p(\Omega)}$ as a candidate, we conclude from estimate (1.1) that

Corollary 1.4. *Let $\Omega \subset \mathbb{C}^n$ be a bounded domain with $\alpha(\Omega) > 0$. For every $p < 2 + \frac{2\alpha(\Omega)}{2n-\alpha(\Omega)}$, one has*

$$K_{\Omega,p}(z) \geq C_{\alpha,p} \sqrt{K_{\Omega}(z)} |\mu(z)|^{\frac{(p-2)n}{p\alpha}}.$$

Remark. *If Ω is a bounded pseudoconvex domain with C^2 -boundary, then $K_{\Omega}(z) \geq C\delta(z)^{-2}$ in view of the Ohsawa-Takegoshi extension theorem [44]. On the other hand, Hopf's lemma implies $|\varrho| \geq C\delta$. Thus*

$$K_{\Omega,p}(z) \geq C_{\alpha,p} \delta(z)^{-(1-\frac{(p-2)n}{p\alpha})} |\log \delta(z)|^{-\frac{(p-2)n}{p\alpha}}$$

as $z \rightarrow \partial\Omega$. Notice also that $\frac{(p-2)n}{p\alpha} < \frac{1}{2} \iff p < 2 + \frac{2\alpha(\Omega)}{2n-\alpha(\Omega)}$.

We would like to mention an interesting connection between Problem 1 and regularity problem of biholomorphic maps. The starting point is the following result of Lempert:

Theorem 1.5 (cf. [39], Theorem 6.2). *Let $\Omega_1 \subset \mathbb{C}^n$ be a bounded domain with C^2 -boundary such that its Bergman projection P_{Ω_1} maps $C_0^{\infty}(\Omega_1)$ into $L^p(\Omega_1)$ for some $p > 2$. Let $\Omega_2 \subset \mathbb{C}^n$ be a bounded domain with real-analytic boundary. Then any biholomorphic map $F : \Omega_1 \rightarrow \Omega_2$ extends to a Hölder continuous map $\overline{\Omega}_1 \rightarrow \overline{\Omega}_2$.*

Notice that if Ω is a domain with $\int_{\Omega} |K_{\Omega}(\cdot, w)|^p$ locally uniformly bounded in w for some $p \geq 1$, then for any $\phi \in C_0^{\infty}(\Omega)$,

$$|P_{\Omega}(\phi)(z)|^p \leq \int_{\zeta \in \text{supp } \phi} |K_{\Omega}(\zeta, z)|^p \|\phi\|_{L^q(\Omega)}^p, \quad (1/p + 1/q = 1),$$

so that

$$(1.2) \quad \int_{z \in \Omega} |P_{\Omega}(\phi)(z)|^p \leq \|\phi\|_{L^q(\Omega)}^p \int_{\zeta \in \text{supp } \phi} \int_{z \in \Omega} |K_{\Omega}(z, \zeta)|^p < \infty,$$

i.e., P_{Ω} maps $C_0^{\infty}(\Omega)$ into $L^p(\Omega)$. Thus we have

Corollary 1.6. *Let $\Omega_1 \subset \mathbb{C}^n$ be a bounded domain with C^2 -boundary such that the integral $\int_{\Omega} |K_{\Omega}(\cdot, w)|^p$ is locally uniformly bounded in w for some $p > 2$. Let $\Omega_2 \subset \mathbb{C}^n$ be a bounded domain with real-analytic boundary. Then any biholomorphic map $F : \Omega_1 \rightarrow \Omega_2$ extends to a Hölder continuous map $\overline{\Omega}_1 \rightarrow \overline{\Omega}_2$.*

In particular, it follows from Corollary 1.6 and Theorem 1.1 that any biholomorphic map between a bounded *pseudoconvex* domain with C^2 -boundary and a bounded domain with real-analytic boundary extends to a Hölder continuous map between their closures, which was first proved in [27].

Conjecture 1.7. *If Ω is a bounded domain with C^2 -boundary, then $\int_{\Omega} |K_{\Omega}(\cdot, w)|^p$ is locally uniformly bounded in w for some $p > 2$.*

With the help of an elegant technique due to Blocki [9] (see also [34] for prior related techniques) on estimating the pluricomplex Green function, we may prove the following

Theorem 1.8. *Let $\Omega \subset \mathbb{C}^n$ be a bounded domain with $\alpha(\Omega) > 0$. For every $0 < r < 1$, there exists a constant $C > 0$ such that*

$$(1.3) \quad \mathcal{B}_{\Omega}(z, w) := \frac{|K_{\Omega}(z, w)|^2}{K_{\Omega}(z)K_{\Omega}(w)} \leq C \left(\min \left\{ \frac{\nu(z)}{\mu(w)}, \frac{\nu(w)}{\mu(z)} \right\} \right)^r, \quad z, w \in \Omega,$$

where $\mu := |\varrho|/(1 + |\log |\varrho||)$ and $\nu := |\varrho|(1 + |\log |\varrho||)^n$.

We call $\mathcal{B}_{\Omega}(z, w)$ the normalized Bergman kernel of Ω . There is a long list of papers concerning point-wise estimates of the *weighted* normalized Bergman kernel $\mathcal{B}_{\Omega, \varphi}(z, w) := \frac{|K_{\Omega, \varphi}(z, w)|^2}{K_{\Omega, \varphi}(z)K_{\Omega, \varphi}(w)}$ when Ω is \mathbb{C}^n or a compact algebraic manifold, after a seminal paper of Christ [22] (see [25], [40], [41], [23], [54], etc.). Quantitative measurements of positivity of $i\partial\bar{\partial}\varphi$ play a crucial role in these works.

The basic difference between $\mathcal{B}_{\Omega}(z, w)$ and $\mathcal{B}_{\Omega, \varphi}(z, w)$ is that the former is always a *biholomorphic invariant*. Skwarczyński [50] showed that

$$d_S(z, w) := \left(1 - \sqrt{\mathcal{B}_{\Omega}(z, w)} \right)^{1/2}$$

gives an invariant distance on a bounded domain Ω . The relationship between d_S and the Bergman distance d_B is as follows

$$(1.4) \quad d_B(z, w) \geq \sqrt{2} d_S(z, w)$$

(see e.g. [36], Corollary 6.4.7). By Theorem 1.8 and (1.4), we may prove the following

Corollary 1.9. *If Ω is a bounded domain with $\alpha(\Omega) > 0$, then for fixed $z_0 \in \Omega$ there exists a constant $C > 0$ such that*

$$(1.5) \quad d_B(z_0, z) \geq C \frac{|\log \delta(z)|}{\log |\log \delta(z)|}$$

provided z sufficiently close to $\partial\Omega$.

Blocki [9] first proved (1.5) for any bounded domain which admits a continuous negative psh function ρ with $C_1\delta^{\alpha} \leq -\rho \leq C_2\delta^{\alpha}$ for some constants $C_1, C_2, \alpha > 0$ (e.g. Ω is a pseudoconvex domain with Lipschitz boundary [31]). Diederich-Ohsawa [28] proved earlier that the following weaker inequality

$$d_B(z_0, z) \geq C \log |\log \delta(z)|$$

holds for more general bounded domains admitting a continuous negative psh function ρ with $C_1\delta^{1/\alpha} \leq -\rho \leq C_2\delta^\alpha$ for some constants $C_1, C_2, \alpha > 0$.

In order to study isometric imbedding of Kähler manifolds, Calabi [13] introduced the notion "diastasis". In [4], Marcel Berger wrote: *It seems to me that the notion of diastasis should make a comeback [...]. For example, it would be interesting to compare the diastasis with the various types of Kobayashi metrics (when they exist).*

Notice that the diastasis $D_B(z, w)$ with respect to the Bergman metric is $-\log \mathcal{B}_\Omega(z, w)$.

Corollary 1.10. *If Ω is a bounded domain with $\alpha(\Omega) > 0$, then for fixed $z_0 \in \Omega$ there exists a constant $C > 0$ such that*

$$(1.6) \quad D_B(z_0, z) \geq Cd_K(z_0, z)$$

where d_K denotes the Kobayashi distance.

Problem 3. *Does one have $d_B(z_0, z) \geq Cd_K(z_0, z)$ for bounded domains with $\alpha(\Omega) > 0$?*

2. L^2 BOUNDARY DECAY ESTIMATES OF THE BERGMAN KERNEL

Proposition 2.1. *Let $\Omega \subset \mathbb{C}^n$ be a pseudoconvex domain. Let ρ be a negative continuous psh function on Ω . Set*

$$\Omega_t = \{z \in \Omega : -\rho(z) > t\}, \quad t > 0.$$

Let $a > 0$ be given. For every $0 < r < 1$, there exist constants $\varepsilon_r, C_r > 0$ such that

$$\int_{-\rho \leq \varepsilon} |K_\Omega(\cdot, w)|^2 \leq C_r K_{\Omega_a}(w)(\varepsilon/a)^r$$

for all $w \in \Omega_a$ and $\varepsilon \leq \varepsilon_r a$.

The proof of the proposition is essentially same as Proposition 6.1 in [20]. For the sake of completeness, we include a proof here. The key ingredient is the following weighted estimate of the L^2 -minimal solution of the $\bar{\partial}$ -equation due to Berndtsson:

Theorem 2.2 (cf. [20], Corollary 2.3). *Let Ω be a bounded pseudoconvex domain in \mathbb{C}^n and $\varphi \in PSH(\Omega)$. Let ψ be a continuous psh function on Ω which satisfies $ri\partial\bar{\partial}\psi \geq i\partial\psi \wedge \bar{\partial}\psi$ as currents for some $0 < r < 1$. Suppose v is a $\bar{\partial}$ -closed $(0, 1)$ -form on Ω such that $\int_\Omega |v|^2 e^{-\varphi} < \infty$. Then the $L^2(\Omega, \varphi)$ -minimal solution of $\bar{\partial}u = v$ satisfies*

$$(2.1) \quad \int_\Omega |u|^2 e^{-\psi-\varphi} \leq \frac{1}{1-r} \int_\Omega |v|_{i\partial\bar{\partial}\psi}^2 e^{-\psi-\varphi}.$$

Here $|v|_{i\partial\bar{\partial}\psi}^2$ should be understood as the infimum of non-negative locally bounded functions H satisfying

$$i\bar{v} \wedge v \leq Hi\partial\bar{\partial}\psi$$

as currents.

Proof of Proposition 2.1. Assume first that Ω is bounded. Let $\kappa : \mathbb{R} \rightarrow [0, 1]$ be a smooth cut-off function such that $\kappa|_{(-\infty, 1]} = 1$, $\kappa|_{[3/2, \infty)} = 0$ and $|\kappa'| \leq 2$. We then have

$$\int_{-\rho \leq \varepsilon} |K_\Omega(\cdot, w)|^2 \leq \int_\Omega \kappa(-\rho/\varepsilon) |K_\Omega(\cdot, w)|^2.$$

By the well-known property of the Bergman projection, we obtain

$$\int_\Omega \kappa(-\rho/\varepsilon) K_\Omega(\cdot, w) \cdot \overline{K_\Omega(\cdot, \zeta)} = \kappa(-\rho(\zeta)/\varepsilon) K_\Omega(\zeta, w) - u(\zeta), \quad \zeta \in \Omega,$$

where u is the $L^2(\Omega)$ -minimal solution of the equation

$$\bar{\partial}u = \bar{\partial}(\kappa(-\rho/\varepsilon)K_\Omega(\cdot, w)) =: v.$$

Since $\kappa(-\rho(w)/\varepsilon) = 0$ provided $\frac{3}{2}\varepsilon \leq a$ (i.e. $\varepsilon \leq 2a/3$), so we have

$$(2.2) \quad \int_{-\rho \leq \varepsilon} |K_\Omega(\cdot, w)|^2 \leq -u(w).$$

Set

$$\psi = -r \log(-\rho), \quad 0 < r < 1.$$

Clearly, ψ is psh and satisfies $ri\partial\bar{\partial}\psi \geq i\partial\psi \wedge \bar{\partial}\psi$, so that

$$i\bar{v} \wedge v \leq C_0 r^{-1} |\kappa'(-\rho/\varepsilon)|^2 |K_\Omega(\cdot, w)|^2 i\partial\bar{\partial}\psi$$

for some numerical constant $C_0 > 0$. Thus by Theorem 2.2 we obtain

$$\begin{aligned} \int_{\Omega} |u|^2 e^{-\psi} &\leq C_r \int_{\varepsilon \leq -\rho \leq \frac{3}{2}\varepsilon} |K_\Omega(\cdot, w)|^2 e^{-\psi} \\ &\leq C_r \varepsilon^r \int_{-\rho \leq \frac{3}{2}\varepsilon} |K_\Omega(\cdot, w)|^2. \end{aligned}$$

Since $e^{-\psi} \geq a^r$ on Ω_a and u is holomorphic there, it follows that

$$\begin{aligned} |u(w)|^2 &\leq K_{\Omega_a}(w) \int_{\Omega_a} |u|^2 \\ &\leq K_{\Omega_a}(w) a^{-r} \int_{\Omega} |u|^2 e^{-\psi} \\ &\leq C_r K_{\Omega_a}(w) (\varepsilon/a)^r \int_{-\rho \leq \frac{3}{2}\varepsilon} |K_\Omega(\cdot, w)|^2. \end{aligned}$$

Thus by (2.2), we obtain

$$\int_{-\rho \leq \varepsilon} |K_\Omega(\cdot, w)|^2 \leq C_r K_{\Omega_a}(w)^{1/2} (\varepsilon/a)^{r/2} \left(\int_{-\rho \leq \frac{3}{2}\varepsilon} |K_\Omega(\cdot, w)|^2 \right)^{1/2}.$$

Notice that

$$\int_{-\rho \leq \frac{3}{2}\varepsilon} |K_\Omega(\cdot, w)|^2 \leq \int_{\Omega} |K_\Omega(\cdot, w)|^2 = K_\Omega(w) \leq K_{\Omega_a}(w)$$

provided $\frac{3}{2}\varepsilon \leq a$. Thus

$$\int_{-\rho \leq \varepsilon} |K_\Omega(\cdot, w)|^2 \leq C_r K_{\Omega_a}(w) (\varepsilon/a)^{r/2}.$$

Replacing ε by $\frac{3}{2}\varepsilon$ in the argument above, we obtain

$$\int_{-\rho \leq \frac{3}{2}\varepsilon} |K_\Omega(\cdot, w)|^2 \leq C_r K_{\Omega_a}(w) (3/2)^{r/2} (\varepsilon/a)^{r/2}$$

provided $(3/2)^2 \varepsilon \leq a$. Thus we may improve the upper bound by

$$\int_{-\rho \leq \varepsilon} |K_\Omega(\cdot, w)|^2 \leq C_r K_{\Omega_a}(w) (\varepsilon/a)^{r/2+r/4}.$$

By induction, we conclude that for every $k \in \mathbb{Z}^+$,

$$\int_{-\rho \leq \varepsilon} |K_\Omega(\cdot, w)|^2 \leq C_{r,k} K_{\Omega_a}(w) (\varepsilon/a)^{r/2+r/4+\dots+r/2^k}$$

provided $(3/2)^k \varepsilon \leq a$. Since $r/2 + r/4 + \dots + r/2^k \rightarrow 1$ as $k \rightarrow \infty$ and $r \rightarrow 1$, we get the desired estimate under the assumption that Ω is bounded.

In general, Ω may be exhausted by an increasing sequence $\{\Omega_j\}$ of bounded pseudoconvex domains. From the argument above we know that

$$\int_{\Omega_j \cap \{-\rho \leq \varepsilon\}} |K_{\Omega_j}(\cdot, w)|^2 \leq C_r K_{\Omega_j \cap \Omega_a}(w) (\varepsilon/a)^r$$

holds for all $j \gg 1$. Since $\Omega_j \uparrow \Omega$, it is well-known that $K_{\Omega_j}(\cdot, w) \rightarrow K_\Omega(\cdot, w)$ locally uniformly in Ω and $K_{\Omega_j \cap \Omega_a}(w) \rightarrow K_{\Omega_a}(w)$. It follows from Fatou's lemma that

$$\begin{aligned} \int_{-\rho \leq \varepsilon} |K_\Omega(\cdot, w)|^2 &= \liminf_{j \rightarrow \infty} \int_{\Omega_j \cap \{-\rho \leq \varepsilon\}} |K_{\Omega_j}(\cdot, w)|^2 \\ &\leq C_r K_{\Omega_a}(w) (\varepsilon/a)^r. \end{aligned}$$

□

Remark. Berndtsson-Charpentier [5] showed that if $\int_\Omega |f|^2 |\rho|^{-r} < \infty$ for some $0 < r < 1$, then

$$\int_\Omega |P_\Omega(f)|^2 |\rho|^{-r} \leq C_r \int_\Omega |f|^2 |\rho|^{-r} < \infty$$

where $P_\Omega(f) := \int_\Omega K_\Omega(z, \cdot) f(\cdot)$ is the Bergman projection. Since $K_\Omega(z, w) = P_\Omega(\phi_w)(z)$ where $\phi_w \in C_0^\infty(\Omega)$ is radially symmetric w.r.t. the center w and satisfies $\int_\Omega \phi_w = 1$, it follows that

$$\int_\Omega |K_\Omega(\cdot, w)|^2 |\rho|^{-r} \leq C_r \int_\Omega |\phi_w|^2 |\rho|^{-r},$$

from which one immediately obtains a somewhat weaker estimate:

$$\int_{-\rho \leq \varepsilon} |K_\Omega(\cdot, w)|^2 \leq C_r \varepsilon^r \int_\Omega |\phi_w|^2 |\rho|^{-r}.$$

Let ϱ be the relative extremal function of a (fixed) closed ball $\overline{B} \subset \Omega$. We have

Proposition 2.3. *Let $\Omega \subset \mathbb{C}^n$ be a bounded domain with $\alpha(\Omega) > 0$. For every $0 < r < 1$, there exist constants $\varepsilon_r, C_r > 0$ such that*

$$(2.3) \quad \int_{-\varrho \leq \varepsilon} |K_\Omega(\cdot, w)|^2 / K_\Omega(w) \leq C_r (\varepsilon / \mu(w))^r$$

for all $\varepsilon \leq \varepsilon_r \mu(w)$, where $\mu = |\varrho|(1 + |\log |\varrho||)^{-1}$.

In order to prove this proposition, we need an elementary estimate of the pluricomplex Green function. Recall that the pluricomplex Green function $g_\Omega(z, w)$ of a domain $\Omega \subset \mathbb{C}^n$ is defined as

$$g_\Omega(z, w) = \sup \{ u(z) : u \in PSH^-(\Omega), u(z) \leq \log |z - w| + O(1) \text{ near } w \}.$$

We first show the following quasi-Hölder-continuity of ϱ :

Lemma 2.4. *Let $\Omega \subset \mathbb{C}^n$ be a bounded domain with $\alpha(\Omega) > 0$. For every $r > 1$ and $0 < \alpha < \alpha(\Omega)$, there exists a constant $C > 0$ such that*

$$(2.4) \quad \varrho(z_2) \geq r \varrho(z_1) - C |z_1 - z_2|^\alpha, \quad z_1, z_2 \in \Omega.$$

Proof. Choose $\rho \in C(\Omega) \cap PSH^-(\Omega)$ with $-\rho \leq C_\alpha \delta^\alpha$. Clearly we have

$$\varrho(z) \geq \frac{\rho(z)}{\inf_{\overline{B}} |\rho|} \geq -C_\alpha \delta^\alpha.$$

To get (2.4), we employ a well-known technique of Walsh [53] as follows. Set $\varepsilon := |z_1 - z_2|$, $\Omega' := \Omega - (z_1 - z_2)$ and

$$u(z) = \begin{cases} \varrho(z) & \text{if } z \in \Omega \setminus \Omega' \\ \max\{\varrho(z), r\varrho(z + z_1 - z_2) - C\varepsilon^\alpha\} & \text{if } z \in \Omega \cap \Omega'. \end{cases}$$

We claim that $u \in PSH^-(\Omega)$ provided $C \gg 1$. Indeed, if $z \in \Omega \cap \partial\Omega'$ then $\delta(z) \leq \varepsilon$, so that

$$\varrho(z) \geq -C_\alpha \delta(z)^\alpha \geq -C_\alpha \varepsilon^\alpha \geq r\varrho(z + z_1 - z_2) - C_\alpha \varepsilon^\alpha.$$

Moreover, if $\varepsilon \leq \varepsilon_r \ll 1$ then $\varrho(z + z_1 - z_2) \leq -1/r$ for $z \in \overline{B}$ since ϱ is continuous on $\overline{\Omega}$. Thus $u|_{\overline{B}} \leq -1$. Since $z_2 = z_1 - (z_1 - z_2) \in \Omega \cap \Omega'$, it follows that

$$\varrho(z_2) \geq u(z_2) \geq r\varrho(z_1) - C_\alpha \varepsilon^\alpha.$$

If $\varepsilon = |z_1 - z_2| > \varepsilon_r$, then (2.4) trivially holds. \square

Remark. *It is not known whether ϱ is Hölder continuous on $\overline{\Omega}$. The answer is positive if $n = 1$ (see [15], p. 138).*

Proposition 2.5. *Let $\Omega \subset \mathbb{C}^n$ be a bounded domain with $\alpha(\Omega) > 0$. There exists a constant $C \gg 1$ such that*

$$(2.5) \quad \{g_\Omega(\cdot, w) < -1\} \subset \{\varrho < -C^{-1}\mu(w)\}, \quad w \in \Omega.$$

Proof. Fix $0 < \alpha < \alpha(\Omega)$. We have $-\varrho \leq C_\alpha \delta^\alpha$ for some constant $C_\alpha > 0$. Clearly, it suffices to consider the case when $|\varrho(w)| \leq 1/2$. Applying Lemma 2.4 with $r = 3/2$, we see that if $\varrho(z) = \varrho(w)/2$ then

$$C_1 |z - w|^\alpha \geq \frac{3}{2} \varrho(z) - \varrho(w) = -\frac{1}{4} \varrho(w),$$

so that

$$\log \frac{|z - w|}{R} \geq \frac{1}{\alpha} \log |\varrho(w)| / (4C_1) - \log R \geq C_2 \log |\varrho(w)|$$

for some constant $C_2 \gg 1$. It follows that

$$\psi(z) := \begin{cases} \log \frac{|z - w|}{R} & \text{if } \varrho(z) \leq \varrho(w)/2 \\ \max \left\{ \log \frac{|z - w|}{R}, 2C_2(\varrho(w)^{-1} \log |\varrho(w)|) \varrho(z) \right\} & \text{otherwise.} \end{cases}$$

is a well-defined negative psh function on Ω with a logarithmic pole at w , and if $\varrho(z) \geq \varrho(w)/2$, then

$$(2.6) \quad g_\Omega(z, w) \geq \psi(z) \geq 2C_2(\varrho(w)^{-1} \log |\varrho(w)|) \varrho(z).$$

Thus

$$\{g_\Omega(\cdot, w) < -1\} \cap \{\varrho \geq \varrho(w)/2\} \subset \{\varrho < -C^{-1}\mu(w)\}$$

provided $C \gg 1$. Since $\{\varrho < \varrho(w)/2\} \subset \{\varrho < -C^{-1}\mu(w)\}$ if $C \gg 1$, we conclude the proof. \square

Proof of Proposition 2.3. Set $A_w := \{g_\Omega(\cdot, w) < -1\}$. It is known from [33] or [19] that

$$(2.7) \quad K_{A_w}(w) \leq C_n K_\Omega(w).$$

By Proposition 2.5, we have

$$(2.8) \quad A_w \subset \Omega_{a(w)} := \{\varrho < -a(w)\}$$

where $a(w) := C^{-1}\mu(w)$ with $C \gg 1$. If we choose $\rho = \varrho$ in Proposition 2.1, it follows that for every $\varepsilon \leq \varepsilon_r a(w)$,

$$(2.9) \quad \begin{aligned} \int_{-\varrho \leq \varepsilon} |K_\Omega(\cdot, w)|^2 &\leq C_r K_{\Omega_{a(w)}}(w) (\varepsilon/a(w))^r \\ &\leq C_{n,r} K_\Omega(w) (\varepsilon/a(w))^r \end{aligned}$$

in view of (2.7), (2.8). \square

3. L^p -INTEGRABILITY OF THE BERGMAN KERNEL

Proof of Theorem 1.1. Without loss of generality, we may assume $\alpha(\Omega) > 0$. For every $0 < \alpha < \alpha(\Omega)$, we may choose $\rho \in PSH^-(\Omega)$ such that

$$-\rho \leq C_\alpha \delta^\alpha$$

for some constant $C_\alpha > 0$. Let S be a compact set in Ω and let $w \in S$. By virtue of Proposition 2.1, we conclude that for every $0 < r < 1$,

$$\int_{-\rho \leq \varepsilon} |K_\Omega(\cdot, w)|^2 \leq C \varepsilon^r$$

where $C = C(n, r, \alpha, S) > 0$. Since $\{\delta \leq \varepsilon\} \subset \{-\rho \leq C_\alpha \varepsilon^\alpha\}$, it follows that

$$\int_{\delta \leq \varepsilon} |K_\Omega(\cdot, w)|^2 \leq C \varepsilon^{r\alpha}.$$

Since $|\delta(\zeta) - \delta(z)| \leq |\zeta - z|$, we have $B(z, \delta(z)) \subset \{\delta \leq 2\delta(z)\}$. By the mean value inequality, we get

$$(3.1) \quad |K_\Omega(z, w)|^2 \leq C_n \delta(z)^{-2n} \int_{\delta \leq 2\delta(z)} |K_\Omega(\cdot, w)|^2 \leq C \delta(z)^{r\alpha - 2n}.$$

Thus for every $\tau > 0$ we have

$$\begin{aligned} \int_\Omega |K_\Omega(\cdot, w)|^{2+\tau} &= \int_{\delta > 1/2} |K_\Omega(\cdot, w)|^{2+\tau} + \sum_{k=1}^{\infty} \int_{2^{-k-1} < \delta \leq 2^{-k}} |K_\Omega(\cdot, w)|^{2+\tau} \\ &\leq C 2^{n\tau} \int_\Omega |K_\Omega(\cdot, w)|^2 + C \sum_{k=1}^{\infty} 2^{(k+1)\tau(n-r\alpha/2)} \int_{\delta \leq 2^{-k}} |K_\Omega(\cdot, w)|^2 \\ &\leq C + C 2^{\tau(n-r\alpha/2)} \sum_{k=1}^{\infty} 2^{-k(r\alpha + \tau(r\alpha/2 - n))} \\ &< \infty \end{aligned}$$

provided $\tau < \frac{2r\alpha}{2n-r\alpha}$. Since r and α can be arbitrarily close to 1 and $\alpha(\Omega)$ respectively, we conclude the proof of the first statement.

Since $\{\delta \leq \varepsilon\} \subset \{-\varrho \leq C_\alpha \varepsilon^\alpha\}$, it follows from Proposition 2.3 that

$$(3.2) \quad \int_{\delta \leq \varepsilon} |K_\Omega(\cdot, w)|^2 / K_\Omega(w) \leq C_{\alpha,r} (\varepsilon^\alpha / \mu(w))^r$$

provided $\varepsilon^\alpha/\mu(w) \leq \varepsilon_r \ll 1$. For every $z \in \Omega$, we have

$$(3.3) \quad |K_\Omega(z, w)|^2/K_\Omega(w) \leq K_\Omega(z) \leq C_n \delta(z)^{-2n},$$

and if $(2\delta(z))^\alpha \leq \varepsilon_r \mu(w)$,

$$(3.4) \quad \begin{aligned} |K_\Omega(z, w)|^2 &\leq C_n \delta(z)^{-2n} \int_{\delta \leq 2\delta(z)} |K_\Omega(\cdot, w)|^2 \\ &\leq C_{\alpha, r} K_\Omega(w) \mu(w)^{-r} \delta(z)^{\alpha r - 2n}. \end{aligned}$$

For every $\tau < \frac{2r\alpha}{2n-r\alpha}$, we conclude from (3.3) that

$$(3.5) \quad \begin{aligned} &\int_{2\delta \geq (\varepsilon_r \mu(w))^{1/\alpha}} |K_\Omega(\cdot, w)|^{2+\tau} \\ &\leq C_n K_\Omega(w)^{\tau/2} \int_{2\delta \geq (\varepsilon_r \mu(w))^{1/\alpha}} |K_\Omega(\cdot, w)|^2 \delta^{-n\tau} \\ &\leq C_{\alpha, r} \frac{K_\Omega(w)^{\tau/2}}{\mu(w)^{n\tau/\alpha}} \int_\Omega |K_\Omega(\cdot, w)|^2 \\ &\leq C_{\alpha, r} \frac{K_\Omega(w)^{1+\tau/2}}{\mu(w)^{n\tau/\alpha}}. \end{aligned}$$

Now choose $k_w \in \mathbb{Z}^+$ such that $(\varepsilon_r \mu(w))^{1/\alpha} \in (2^{-k_w-1}, 2^{-k_w}]$ (it suffices to consider the case when $\mu(w)$ is sufficiently small). We then have

$$(3.6) \quad \begin{aligned} &\int_{2\delta < (\varepsilon_r \mu(w))^{1/\alpha}} |K_\Omega(\cdot, w)|^{2+\tau} \\ &\leq \sum_{k=k_w}^{\infty} \int_{2^{-k-1} < \delta \leq 2^{-k}} |K_\Omega(\cdot, w)|^{2+\tau} \\ &\leq C_{\alpha, r, \tau} \frac{K_\Omega(w)^{\tau/2}}{\mu(w)^{\tau r/2}} \sum_{k=k_w}^{\infty} 2^{k\tau(n-r\alpha/2)} \int_{\delta \leq 2^{-k}} |K_\Omega(\cdot, w)|^2 \quad (\text{by (3.4)}) \\ &\leq C_{\alpha, r, \tau} \frac{K_\Omega(w)^{1+\tau/2}}{\mu(w)^{r(1+\tau/2)}} \sum_{k=k_w}^{\infty} 2^{-k(r\alpha+\tau(r\alpha/2-n))} \quad (\text{by (3.2)}) \\ &\leq C_{\alpha, r, \tau} \frac{K_\Omega(w)^{1+\tau/2}}{\mu(w)^{r(1+\tau/2)}} \mu(w)^{(r\alpha+\tau(r\alpha/2-n))/\alpha} \\ &\leq C_{\alpha, r, \tau} \frac{K_\Omega(w)^{1+\tau/2}}{\mu(w)^{\tau n/\alpha}}. \end{aligned}$$

By (3.5) and (3.6), (1.1) immediately follows. \square

Proof of Theorem 1.2. It suffices to use the following lemma instead of (3.1) in the proof of the first statement in Theorem 1.1. \square

Lemma 3.1. *Let Ω be a domain in \mathbb{C} . For every compact set $S \subset \Omega$ and $\alpha < \alpha(\Omega)$, there exists a constant $C > 0$ such that*

$$|K_\Omega(z, w)| \leq C \delta(z)^{\alpha-1}, \quad z \in \Omega, w \in S.$$

Proof. Let $g_\Omega(z, w)$ be the (negative) Green function on Ω . Let $\Delta(c, r)$ be the disc with centre c and radius r . Fix $w \in S$ and $z \in \Omega$ for a moment. Clearly, it suffices to consider the case when $\delta(z) \leq \delta(w)/4$. Since $g_\Omega(\xi, \zeta)$ is harmonic in $\xi \in \Delta(z, \delta(z))$ and $\zeta \in \Delta(w, \delta(w)/2)$ respectively, we conclude from Poisson's formula that

$$g_\Omega(\xi, \zeta) = \frac{1}{4\pi^2} \int_0^{2\pi} \int_0^{2\pi} g_\Omega \left(z + \frac{\delta(z)}{2} e^{i\theta}, w + \frac{\delta(w)}{2} e^{i\vartheta} \right) \frac{\frac{\delta(z)^2}{4} - |\xi - z|^2}{\left| \frac{\delta(z)}{2} e^{i\theta} - (\xi - z) \right|^2} \frac{\frac{\delta(w)^2}{4} - |\zeta - w|^2}{\left| \frac{\delta(w)}{2} e^{i\vartheta} - (\zeta - w) \right|^2} d\theta d\vartheta$$

where $\xi \in \Delta(z, \delta(z)/4)$ and $\zeta \in \Delta(w, \delta(w)/4)$. By the extremal property of g_Ω , it is easy to verify $-g_\Omega \leq C\delta(z)^\alpha$ on $\partial\Delta(z, \delta(z)/2) \times \partial\Delta(w, \delta(w)/2)$. Thus

$$\left| \frac{\partial^2 g_\Omega(\xi, \zeta)}{\partial \xi \partial \bar{\zeta}} \right| \leq C\delta(z)^{\alpha-1}.$$

Together with Schiffer's formula $K_\Omega(\xi, \zeta) = \frac{2}{\pi} \frac{\partial^2 g_\Omega(\xi, \zeta)}{\partial \xi \partial \bar{\zeta}}$ (cf. [49]), the assertion immediately follows. \square

In order to prove Proposition 1.3, we need the following

Theorem 3.2 (cf. [14], §6, Theorem 1). *Let $\Omega = \mathbb{C} \setminus E$ where $E \subset \mathbb{C}$ is a compact set. Then*

- (1) $A^2(\Omega) \neq \{0\}$ if and only if $\text{Cap}(E) > 0$.
- (2) $A^p(\Omega) = \{0\}$ if $\Lambda_{2-q}(E) < \infty$, $2 < p < \infty$, $\frac{1}{p} + \frac{1}{q} = 1$. Here $\Lambda_s(E)$ denotes the s -dimensional Hausdorff measure of E .

Remark. Let $\Omega \subset \mathbb{C}$ be a domain and E a closed polar set in Ω . It is well-known that E is removable for negative harmonic functions, so that $g_{\Omega \setminus E}(z, w) = g_\Omega(z, w)$ for $z, w \in \Omega \setminus E$. Thus $K_{\Omega \setminus E}(z, w) = K_\Omega(z, w)$ in view of Schiffer's formula. By the reproducing property of the Bergman kernel, we immediately get the known fact that $A^2(\Omega \setminus E) = A^2(\Omega)$.

Proof Proposition 1.3. Suppose on the contrary $\beta(\Omega) > 2 + \frac{\dim_H(E)}{1 - \dim_H(E)}$. Fix

$$\beta(\Omega) > p > 2 + \frac{\dim_H(E)}{1 - \dim_H(E)}$$

and let q be the conjugate exponent of p , i.e., $\frac{1}{p} + \frac{1}{q} = 1$. We then have $K_\Omega(\cdot, w) \in A^p(\Omega)$ for fixed w . Since

$$\dim_H(E) = \sup\{s : \Lambda_s(E) = \infty\}$$

and $2 - q > \dim_H(E)$, it follows that $\Lambda_{2-q}(E) < \infty$, so that $K_\Omega(\cdot, w) = 0$ in view of Theorem 3.2/(2). On the other side, since $\text{Cap}(E) > 0$, so $K_\Omega(\cdot, w) \neq 0$ in view of Theorem 3.2/(1), which is absurd. \square

Theorem 1.2 implies $\beta(\Omega) \rightarrow \infty$ as $\alpha(\Omega) \rightarrow 1$ for planar domains (notice that $\alpha(\Omega) = 1$ when $\Omega \subset \mathbb{C}$ is convex or $\partial\Omega$ is C^1). It is also known that $\beta(\Omega) = \infty$ if Ω is a bounded smooth convex domain in \mathbb{C}^n (cf. [11]). Thus it is reasonable to make the following

Conjecture 3.3. *If $\Omega \subset \mathbb{C}^n$ is convex, then $\beta(\Omega) = \infty$.*

4. APPLICATIONS OF L^p -INTEGRABILITY OF THE BERGMAN KERNEL

We first study density of $A^p(\Omega) \cap A^2(\Omega)$ in $A^2(\Omega)$.

Proposition 4.1. *Let Ω be a pseudoconvex domain in \mathbb{C}^n . For every $1 \leq p < 2 + \frac{2\alpha(\Omega)}{2n-\alpha(\Omega)}$, $A^p(\Omega) \cap A^2(\Omega)$ lies dense in $A^2(\Omega)$.*

Proof. Choose a sequence of functions $\chi_j \in C_0^\infty(\Omega)$ such that $0 \leq \chi_j \leq 1$ and the sequence of sets $\{\chi_j = 1\}$ exhausts Ω . Given $f \in A^2(\Omega)$, we set $f_j = P_\Omega(\chi_j f)$. Clearly, $f_j \in A^p(\Omega) \cap A^2(\Omega)$ in view of Theorem 1.1 and (1.2). Moreover,

$$\|f_j - f\|_{L^2(\Omega)} = \|P_\Omega((\chi_j - 1)f)\|_{L^2(\Omega)} \leq \|(\chi_j - 1)f\|_{L^2(\Omega)} \rightarrow 0.$$

□

Similarly, we may prove the following

Proposition 4.2. *Let Ω be a domain in \mathbb{C} . For every $1 \leq p < 2 + \frac{\alpha(\Omega)}{1-\alpha(\Omega)}$, $A^p(\Omega) \cap A^2(\Omega)$ lies dense in $A^2(\Omega)$.*

Next we study reproducing property of the Bergman kernel in $A^p(\Omega)$.

Proposition 4.3. *Let Ω be a bounded domain in \mathbb{C} with $\alpha(\Omega) > 0$. If $p > 2 - \alpha(\Omega)$, then $f = P_\Omega(f)$ for all $f \in A^p(\Omega)$.*

Proof. Suppose $f \in A^p(\Omega)$ with $p > 2 - \alpha(\Omega)$. Let q be the conjugate exponent of p . Since $q < 2 + \frac{\alpha(\Omega)}{1-\alpha(\Omega)}$, so the integral $\int_\Omega f(\cdot) K_\Omega(z, \cdot)$ is well-defined in view of Theorem 1.2. Clearly, it suffices to consider the case $p < 2$. By Theorem 1 of Hedberg [32], we may find a sequence $f_j \in \mathcal{O}(\overline{\Omega}) \subset A^2(\Omega) \subset A^p(\Omega)$ such that $\|f_j - f\|_{L^p(\Omega)} \rightarrow 0$. It follows that for every $z \in \Omega$,

$$f(z) = \lim_{j \rightarrow \infty} f_j(z) = \lim_{j \rightarrow \infty} \int_\Omega f_j(\cdot) K_\Omega(z, \cdot) = \int_\Omega f(\cdot) K_\Omega(z, \cdot)$$

since $K_\Omega(z, \cdot) \in L^q(\Omega)$.

□

For a bounded domain $\Omega \subset \mathbb{C}^n$, the Berezin transform T_Ω of Ω is defined as

$$T_\Omega(f)(z) = \int_\Omega f(\cdot) \frac{|K_\Omega(\cdot, z)|^2}{K_\Omega(z)}, \quad z \in \Omega, f \in L^\infty(\Omega).$$

Clearly, one has $f = T_\Omega(f)$ for all $f \in A^\infty(\Omega)$.

Corollary 4.4. *Let Ω be a bounded domain in \mathbb{C} with $\alpha(\Omega) > 0$. If $p > 2/\alpha(\Omega) - 1$, then $f = T_\Omega(f)$ for all $f \in A^p(\Omega)$.*

Proof. Set $p' = \frac{2p}{p+1}$. It follows from Hölder's inequality that

$$\begin{aligned} \int_\Omega |f K_\Omega(\cdot, z)|^{p'} &\leq \left(\int_\Omega |f|^{\frac{p'}{2-p'}} \right)^{2-p'} \left(\int_\Omega |K_\Omega(\cdot, z)|^{\frac{p'}{p'-1}} \right)^{p'-1} \\ &= \left(\int_\Omega |f|^p \right)^{2-p'} \left(\int_\Omega |K_\Omega(\cdot, z)|^{\frac{p'}{p'-1}} \right)^{p'-1} \\ &< \infty, \end{aligned}$$

since $p' > 2 - \alpha(\Omega)$ and $\frac{p'}{p'-1} < 2 + \frac{\alpha(\Omega)}{1-\alpha(\Omega)}$. Thus $h := fK_\Omega(\cdot, z)/K_\Omega(z) \in A^{p'}(\Omega)$ for fixed $z \in \Omega$, so that

$$f(z) = h(z) = \int_{\Omega} h(\cdot)K_\Omega(z, \cdot) = \int_{\Omega} f(\cdot) \frac{|K_\Omega(\cdot, z)|^2}{K_\Omega(z)}.$$

□

For higher-dimensional cases, we can only prove the following

Proposition 4.5. *Let Ω be a bounded pseudoconvex domain in \mathbb{C}^n . Suppose there exists a negative psh exhaustion function ρ on Ω such that for suitable constants $C, \alpha > 0$,*

$$|\rho(z) - \rho(w)| \leq C|z - w|^\alpha, \quad z, w \in \Omega.$$

For every $p > \frac{4n}{2n+\alpha}$, one has $f = P_\Omega(f)$ for all $f \in A^p(\Omega)$.

Proof. Set $\Omega_t = \{-\rho > t\}$, $t \geq 0$, and $\rho_t := \rho + t$. For every $z \in \Omega_t$, we choose $z^* \in \partial\Omega_t$ such that $|z - z^*| = \delta_t(z) := d(z, \partial\Omega_t)$. We then have

$$|\rho_t(z)| = |\rho_t(z) - \rho_t(z^*)| \leq C|z - z^*|^\alpha = C\delta_t(z)^\alpha$$

where C is a constant independent of t . By a similar argument as the proof of Theorem 1.1, we may show that for fixed $w \in \Omega$,

$$\int_{\Omega_t} |K_{\Omega_t}(\cdot, w)|^q \leq C = C(q, w) < \infty$$

holds uniformly in $t \ll 1$ for every $q < 2 + \frac{2\alpha}{2n-\alpha}$. Let $2 > p > \frac{4n}{2n+\alpha}$ and $f \in A^p(\Omega)$. Fix $z \in \Omega$ for a moment. For every $t \ll 1$, we have $z \in \Omega_t$ and

$$(4.1) \quad f(z) = \int_{\Omega_t} f(\cdot)K_{\Omega_t}(z, \cdot).$$

Notice that

$$\begin{aligned} & \left| \int_{\Omega} f(\cdot)K_{\Omega}(z, \cdot) - \int_{\Omega_t} f(\cdot)K_{\Omega_t}(z, \cdot) \right| \\ & \leq \int_{\Omega_t} |f||K_{\Omega}(z, \cdot) - K_{\Omega_t}(z, \cdot)| + \int_{\Omega \setminus \Omega_t} |f||K_{\Omega}(z, \cdot)| \\ (4.2) \quad & \leq \|f\|_{L^p(\Omega)} \|K_{\Omega}(z, \cdot) - K_{\Omega_t}(z, \cdot)\|_{L^q(\Omega_t)} + \|f\|_{L^p(\Omega \setminus \Omega_t)} \|K_{\Omega}(z, \cdot)\|_{L^q(\Omega)} \end{aligned}$$

where $\frac{1}{p} + \frac{1}{q} = 1$ (which implies $q < 2 + \frac{2\alpha}{2n-\alpha}$). Take $0 < \gamma \ll 1$ so that $\frac{q-\gamma}{1-\gamma/2} < 2 + \frac{2\alpha}{2n-\alpha}$. We then have

$$\begin{aligned} & \int_{\Omega_t} |K_{\Omega}(z, \cdot) - K_{\Omega_t}(z, \cdot)|^q \\ & = \int_{\Omega_t} |K_{\Omega}(z, \cdot) - K_{\Omega_t}(z, \cdot)|^\gamma |K_{\Omega}(z, \cdot) - K_{\Omega_t}(z, \cdot)|^{q-\gamma} \\ & \leq \left(\int_{\Omega_t} |K_{\Omega}(z, \cdot) - K_{\Omega_t}(z, \cdot)|^2 \right)^{\gamma/2} \left(\int_{\Omega_t} |K_{\Omega}(z, \cdot) - K_{\Omega_t}(z, \cdot)|^{\frac{q-\gamma}{1-\gamma/2}} \right)^{1-\gamma/2} \end{aligned}$$

in view of Hölder's inequality. Since

$$\begin{aligned}
& \int_{\Omega_t} |K_{\Omega}(z, \cdot) - K_{\Omega_t}(z, \cdot)|^2 \\
&= \int_{\Omega_t} |K_{\Omega}(z, \cdot)|^2 + \int_{\Omega_t} |K_{\Omega_t}(z, \cdot)|^2 \\
&\quad - 2\operatorname{Re} \int_{\Omega_t} K_{\Omega}(z, \cdot) \overline{K_{\Omega_t}(\cdot, z)} \\
&\leq |K_{\Omega_t}(z) - K_{\Omega}(z)|^2 \\
&\rightarrow 0 \quad (t \rightarrow 0)
\end{aligned}$$

and

$$\begin{aligned}
& \int_{\Omega_t} |K_{\Omega}(z, \cdot) - K_{\Omega_t}(z, \cdot)|^{\frac{q-\gamma}{1-\gamma/2}} \\
&\leq 2^{\frac{q-\gamma}{1-\gamma/2}} \left(\int_{\Omega} |K_{\Omega}(z, \cdot)|^{\frac{q-\gamma}{1-\gamma/2}} + \int_{\Omega_t} |K_{\Omega_t}(z, \cdot)|^{\frac{q-\gamma}{1-\gamma/2}} \right) \\
&\leq C,
\end{aligned}$$

it follows from (4.1) and (4.2) that $f = P_{\Omega}(f)$. \square

Similarly, we have

Corollary 4.6. *If $p > 2n/\alpha$, then $f = T_{\Omega}(f)$ for all $f \in A^p(\Omega)$.*

5. ESTIMATE OF THE PLURICOMPLEX GREEN FUNCTION

The goal of this section is to show the following

Proposition 5.1. *Let $\Omega \subset \mathbb{C}^n$ be a bounded domain with $\alpha(\Omega) > 0$. There exists a constant $C \gg 1$ such that*

$$(5.1) \quad \{g_{\Omega}(\cdot, w) < -1\} \subset \{\varrho > -C\nu(w)\}, \quad w \in \Omega,$$

where $\nu = |\varrho|(1 + |\log |\varrho||)^n$.

We will follow the argument of Blocki [9] with necessary modifications. The key observation is the following

Lemma 5.2 (cf. [9]). *Let $\Omega \subset \mathbb{C}^n$ be a bounded hyperconvex domain. Suppose ζ, w are two points in Ω such that the closed balls $\overline{B}(\zeta, \varepsilon), \overline{B}(w, \varepsilon) \subset \mathbb{C}^n$ and $\overline{B}(\zeta, \varepsilon) \cap \overline{B}(w, \varepsilon) = \emptyset$. Then there exists $\tilde{\zeta} \in \overline{B}(\zeta, \varepsilon)$ such that*

$$(5.2) \quad |g_{\Omega}(\tilde{\zeta}, w)|^n \leq n!(\log R/\varepsilon)^{n-1} |g_{\Omega}(w, \zeta)|$$

where $R := \operatorname{diam}(\Omega)$

For the sake of completeness, we include a proof here, which relies heavily on the following fundamental results.

Theorem 5.3 (cf. [24]). *Let Ω be a bounded hyperconvex domain in \mathbb{C}^n .*

- (1) *For every $w \in \Omega$, one has $(dd^c g_{\Omega}(\cdot, w))^n = (2\pi)^n \delta_w$ where δ_w denotes the Dirac measure at w .*
- (2) *For every $\zeta \in \Omega$ and $\eta > 0$, one has $\int_{\Omega} (dd^c \max\{g_{\Omega}(\cdot, \zeta), -\eta\})^n = (2\pi)^n$.*

Theorem 5.4 (cf. [7], see also [8]). *Let Ω be a bounded domain in \mathbb{C}^n . Assume that $u, v \in PSH^- \cap L^\infty(\Omega)$ are non-positive psh functions such that $u = 0$ on $\partial\Omega$. Then*

$$(5.3) \quad \int_{\Omega} |u|^n (dd^c v)^n \leq n! \|v\|_{\infty}^{n-1} \int_{\Omega} |v| (dd^c u)^n.$$

Proof of Lemma 5.2. Let $\eta = \log R/\varepsilon$. Since $g_{\Omega}(z, \zeta) \geq \log |z - \zeta|/R$, it follows that

$$\{g_{\Omega}(\cdot, \zeta) = -\eta\} \subset \overline{B}(\zeta, \varepsilon).$$

Applying first Theorem 5.4 with $u = \max\{g_{\Omega}(\cdot, w), -t\}$ and $v = \max\{g_{\Omega}(\cdot, \zeta), -\eta\}$ then letting $t \rightarrow +\infty$, we obtain

$$\int_{\Omega} |g_{\Omega}(\cdot, w)|^n (dd^c \max\{g_{\Omega}(\cdot, \zeta), -\eta\})^n \leq n! (2\pi)^n \eta^{n-1} |g_{\Omega}(w, \zeta)|$$

in view of Theorem 5.3/(1). Since $\overline{B}(\zeta, \varepsilon) \cap \overline{B}(w, \varepsilon) = \emptyset$, it follows that g_{Ω} is continuous on $\overline{B}(\zeta, \varepsilon)$, so that there exists $\tilde{\zeta} \in \overline{B}(\zeta, \varepsilon)$ such that

$$|g_{\Omega}(\tilde{\zeta}, w)| = \min_{\overline{B}(\zeta, \varepsilon)} |g_{\Omega}(\cdot, w)|.$$

Since the measure $(dd^c \max\{g_{\Omega}(\cdot, \zeta), -\eta\})^n$ is supported on $\{g_{\Omega}(\cdot, \zeta) = -\eta\}$ with total mass $(2\pi)^n$, we immediately get (5.2). \square

Proof of Proposition 5.1. Clearly, it suffices to consider the case when w is sufficiently close to $\partial\Omega$. Fix $\zeta \in \Omega$ with $\varrho(\zeta) \leq 2\varrho(w)$ for a moment. Set $\varepsilon := |\varrho(w)|^{2/\alpha}$. Since $\varepsilon \leq C_{\alpha}^{2/\alpha} \delta(w)^2$, we see that $\overline{B}(w, \varepsilon) \subset \Omega$ provided $\delta(w) \leq \varepsilon_{\alpha} \ll 1$. For every $z \in \Omega$ with $\delta(z) \leq \varepsilon$, we have

$$(5.4) \quad |\varrho(z)| \leq C_{\alpha} \delta(z)^{\alpha} \leq C_{\alpha} \varepsilon^{\alpha} = C_{\alpha} |\varrho(w)|^2 \quad (\leq |\varrho(w)|/2)$$

provided $\delta(w) \leq \varepsilon_{\alpha} \ll 1$. It follows from (2.6) and (5.4) that for every $\tau > 0$ there exists $\varepsilon_{\tau} \ll \varepsilon_{\alpha}$ such that

$$(5.5) \quad \sup_{\delta \leq \varepsilon} |g_{\Omega}(\cdot, w)| \leq \tau$$

provided $\delta(w) \leq \varepsilon_{\tau}$. Since

$$C_{\alpha} \delta(\zeta)^{\alpha} \geq -\varrho(\zeta) \geq -2\varrho(w) = 2\varepsilon^{\alpha/2}$$

and Lemma 2.4 yields

$$C_1 |\zeta - w|^{\alpha} \geq \frac{3}{2} \varrho(w) - \varrho(\zeta) \geq -\frac{1}{2} \varrho(w) = \frac{1}{2} \varepsilon^{\alpha/2},$$

it follows that if $\delta(w) \leq \varepsilon_{\tau} \ll 1$ then $\overline{B}(\zeta, \varepsilon) \subset \Omega$ and

$$(5.6) \quad \overline{B}(\zeta, \varepsilon) \cap \overline{B}(w, \varepsilon) = \emptyset.$$

By Lemma 5.2, there exists $\tilde{\zeta} \in \overline{B}(\zeta, \varepsilon)$ such that (5.2) holds.

Now set

$$\Psi(z) := \sup\{u(z) : u \in PSH^-(\Omega), u|_{\overline{B}(w, \varepsilon)} \leq -1\}.$$

We claim that

$$(5.7) \quad g_{\Omega}(z, w) \geq \log R/\varepsilon \Psi(z), \quad z \in \Omega \setminus B(w, \varepsilon); \quad g_{\Omega}(z, w) \leq \log \delta(w)/\varepsilon \Psi(z), \quad z \in \Omega.$$

To see this, first notice that

$$(5.8) \quad \log \frac{|z - w|}{R} \leq g_{\Omega}(z, w) \leq \log \frac{|z - w|}{\delta(w)}, \quad z \in \Omega.$$

Since

$$u(z) = \begin{cases} \log |z - w|/R & \text{if } z \in B(w, \varepsilon) \\ \max \{ \log |z - w|/R, \log R/\varepsilon \Psi(z) \} & \text{if } z \in \Omega \setminus B(w, \varepsilon) \end{cases}$$

is a negative psh function on Ω with a logarithmic pole at w , it follows that

$$g_\Omega(z, w) \geq \log R/\varepsilon \Psi(z), \quad z \in \Omega \setminus B(w, \varepsilon).$$

Since (5.8) implies $g_\Omega(\cdot, w)|_{\overline{B}(w, \varepsilon)} \leq \log \varepsilon/\delta(w)$, so we have

$$\Psi(z) \geq \frac{g_\Omega(z, w)}{\log \delta(w)/\varepsilon}, \quad z \in \Omega.$$

By (5.5) and (5.7) we obtain

$$(5.9) \quad \sup_{\delta \leq \varepsilon} |\Psi| \leq \frac{\tau}{\log \delta(w)/\varepsilon}.$$

Set $\tilde{\Omega} = \Omega - (\tilde{\zeta} - \zeta)$ and

$$v(z) = \begin{cases} \Psi(z) & \text{if } z \in \Omega \setminus \tilde{\Omega} \\ \max \left\{ \Psi(z), \Psi(z + \tilde{\zeta} - \zeta) - \frac{\tau}{\log \delta(w)/\varepsilon} \right\} & \text{if } z \in \Omega \cap \tilde{\Omega}. \end{cases}$$

Since $\Omega \cap \partial \tilde{\Omega} \subset \{\delta \leq \varepsilon\}$, it follows from (5.9) that $v \in PSH^-(\Omega)$. Since

$$\Psi(z) \leq \frac{\log |z - w|/\delta(w)}{\log R/\varepsilon}, \quad z \in \Omega \setminus B(w, \varepsilon)$$

in view of (5.8) and (5.7), and $z + \tilde{\zeta} - \zeta \in \overline{B}(w, 2\varepsilon)$ if $z \in \overline{B}(w, \varepsilon)$, it follows from the maximal principle that

$$v|_{\overline{B}(w, \varepsilon)} \leq -\frac{\log \delta(w)/(2\varepsilon)}{\log R/\varepsilon}.$$

Thus

$$\Psi(\tilde{\zeta}) - \frac{\tau}{\log \delta(w)/\varepsilon} \leq v(\zeta) \leq \frac{\log \delta(w)/(2\varepsilon)}{\log R/\varepsilon} \Psi(\zeta).$$

Combining with (5.6) and (5.7), we obtain

$$g_\Omega(\zeta, w) \geq \frac{(\log R/\varepsilon)^2}{\log \delta(w)/\varepsilon \cdot \log \delta(w)/(2\varepsilon)} \left(g_\Omega(\tilde{\zeta}, w) - \tau \right) \geq C_3 \left(g_\Omega(\tilde{\zeta}, w) - \tau \right)$$

since $\delta(w) \geq |\varrho(w)/C_\alpha|^{1/\alpha} = \sqrt{\varepsilon}/C_\alpha^{1/\alpha}$. If we choose $\tau = \frac{1}{2C_3}$, then

$$\begin{aligned} g_\Omega(\zeta, w) &\geq -C_3(n!)^{1/n} (\log R/\varepsilon)^{1-1/n} |g_\Omega(w, \zeta)|^{1/n} - 1/2 \quad (\text{by (5.2)}) \\ &\geq -C_4 |\log |\varrho(w)||^{1-1/n} \frac{|\varrho(w) \log |\varrho(\zeta)||^{1/n}}{|\varrho(\zeta)|^{1/n}} - 1/2 \quad (\text{by (2.6)}) \\ &\geq -C_5 \frac{|\varrho(w)|^{1/n} |\log |\varrho(w)||}{|\varrho(\zeta)|^{1/n}} - 1/2 \end{aligned}$$

since $\varrho(\zeta) \leq 2\varrho(w)$. Thus

$$\{g_\Omega(\cdot, w) < -1\} \cap \{\varrho \leq 2\varrho(w)\} \subset \{\varrho > -C\nu(w)\}$$

provided $C \gg 1$. Since $\{\varrho > 2\varrho(w)\} \subset \{\varrho > -C\nu(w)\}$ if $C \gg 1$, we conclude the proof. \square

6. POINT-WISE ESTIMATE OF THE NORMALIZED BERGMAN KERNEL AND APPLICATIONS

Proof of Theorem 1.8. By Proposition 2.3, we know that for every $0 < r < 1$ there exist constants $\varepsilon_r, C_r > 0$ such that

$$\int_{-\varrho \leq \varepsilon} |K_\Omega(\cdot, w)|^2 / K_\Omega(w) \leq C_r (\varepsilon / \mu(w))^r$$

for all $\varepsilon \leq \varepsilon_r \mu(w)$. Fix $z \in \Omega$ with $b(z) := C\nu(z) \leq \varepsilon_r \mu(w)$ for a moment, where C is the constant in (5.1). Let $\chi : \mathbb{R} \rightarrow [0, 1]$ be a smooth function satisfying $\chi|_{(0, \infty)} = 0$ and $\chi|_{(-\infty, -\log 2)} = 1$. We proceed the proof in a similar way as [19]. Notice that $g_\Omega(\cdot, z)$ is a continuous negative psh function on $\Omega \setminus \{z\}$ which satisfies

$$-i\partial\bar{\partial}\log(-g_\Omega(\cdot, z)) \geq i\partial\log(-g_\Omega(\cdot, z)) \wedge \bar{\partial}\log(-g_\Omega(\cdot, z))$$

as currents. By virtue of the Donnelly-Fefferman estimate (cf. [29], see also [5]), there exists a solution of the equation

$$\bar{\partial}u = K_\Omega(\cdot, w)\bar{\partial}\chi(-\log(-g_\Omega(\cdot, z)))$$

such that

$$\begin{aligned} \int_\Omega |u|^2 e^{-2ng_\Omega(\cdot, z)} &\leq C_0 \int_\Omega |K_\Omega(\cdot, w)|^2 |\bar{\partial}\chi(-\log(-g_\Omega(\cdot, z)))|^2 e^{-i\partial\bar{\partial}\log(-g_\Omega(\cdot, z))} e^{-2ng_\Omega(\cdot, z)} \\ &\leq C_n \int_{\varrho > -b(z)} |K_\Omega(\cdot, w)|^2 \quad (\text{by (5.1)}) \\ &\leq C_{n,r} K_\Omega(w) (\nu(z) / \mu(w))^r. \end{aligned}$$

Set

$$f := K_\Omega(\cdot, w)\chi(-\log(-g_\Omega(\cdot, z))) - u.$$

Clearly, we have $f \in \mathcal{O}(\Omega)$. Since $g_\Omega(\zeta, z) = \log|\zeta - z| + O(1)$ as $\zeta \rightarrow z$ and u is holomorphic in a neighborhood of z , it follows that $u(z) = 0$, i.e. $f(z) = K_\Omega(z, w)$. Moreover, we have

$$\begin{aligned} \int_\Omega |f|^2 &\leq 2 \int_{\varrho > -b(z)} |K_\Omega(\cdot, w)|^2 + 2 \int_\Omega |u|^2 \\ &\leq C_{n,r} K_\Omega(w) (\nu(z) / \mu(w))^r \end{aligned}$$

since $g_\Omega(\cdot, z) < 0$. Thus we get

$$K_\Omega(z) \geq \frac{|f(z)|^2}{\|f\|_{L^2(\Omega)}^2} \geq C_{n,r}^{-1} \frac{|K_\Omega(z, w)|^2}{K_\Omega(w)} (\mu(w) / \nu(z))^r,$$

and

$$\mathcal{B}_\Omega(z, w) \leq C_{n,r} (\nu(z) / \mu(w))^r.$$

If $b(z) > \varepsilon_r \mu(w)$, then the inequality above trivially holds since $\frac{|K_\Omega(z, w)|^2}{K_\Omega(z)K_\Omega(w)} \leq 1$. By symmetry of \mathcal{B}_Ω , the assertion immediately follows. \square

Remark. It would be interesting to get point-wise estimates for $\frac{|S_\Omega(z, w)|^2}{S_\Omega(z)S_\Omega(w)}$ where S_Ω is the Szegő kernel (compare [21]).

Proof of Corollary 1.9. Let $z \in \Omega$ be an arbitrarily fixed point which is sufficiently close to $\partial\Omega$. By the Hopf-Rinow theorem, there exists a Bergman geodesic γ joining z_0 to z , for ds_B^2 is complete on Ω . We may choose a finite number of points $\{z_k\}_{k=1}^m \subset \gamma$ with the following order

$$z_0 \rightarrow z_1 \rightarrow z_2 \rightarrow \cdots \rightarrow z_m \rightarrow z,$$

where

$$|\varrho(z_{k+1})|(1 + |\log |\varrho(z_{k+1})||)^{n+2} = |\varrho(z_k)|$$

and

$$|\varrho(z)|(1 + |\log |\varrho(z)||)^{n+2} \geq |\varrho(z_m)|.$$

Since

$$\begin{aligned} \frac{\nu(z_{k+1})}{\mu(z_k)} &= \frac{|\varrho(z_{k+1})|}{|\varrho(z_k)|} (1 + |\log |\varrho(z_{k+1})||)^n (1 + |\log |\varrho(z_k)||) \\ &\leq \frac{|\varrho(z_{k+1})|}{|\varrho(z_k)|} (1 + |\log |\varrho(z_{k+1})||)^{n+1} \\ &= (1 + |\log |\varrho(z_{k+1})||)^{-1}, \end{aligned}$$

it follows from Theorem 1.8 that there exists $k_0 \in \mathbb{Z}^+$ such that $\mathcal{B}_\Omega(z_k, z_{k+1}) \leq 1/4$ for all $k \geq k_0$. By (1.4), we get

$$d_B(z_k, z_{k+1}) \geq 1.$$

Notice that

$$\begin{aligned} |\varrho(z_{k_0})| &= |\varrho(z_{k_0+1})| |\log |\varrho(z_{k_0+1})||^{n+2} \\ &\leq |\varrho(z_{k_0+2})| |\log |\varrho(z_{k_0+2})||^{2(n+2)} \\ &\leq \cdots \leq |\varrho(z_m)| |\log |\varrho(z_m)||^{(m-k_0)(n+2)}. \end{aligned}$$

Thus we have

$$m - k_0 \geq \text{const.} \frac{|\log |\varrho(z_m)||}{\log |\log |\varrho(z_m)||} \geq \text{const.} \frac{|\log |\varrho(z)||}{\log |\log |\varrho(z)||},$$

so that

$$\begin{aligned} d_B(z, z_0) &\geq \sum_{k=k_0}^{m-1} d_B(z_k, z_{k+1}) \geq m - k_0 - 1 \\ &\geq \text{const.} \frac{|\log |\varrho(z)||}{|\log |\log |\varrho(z)||} \\ &\geq \text{const.} \frac{|\log \delta(z)|}{\log |\log \delta(z)|}, \end{aligned}$$

since $|\varrho(z)| \leq C_\alpha \delta^\alpha$ for any $\alpha < \alpha(\Omega)$. □

Proof of Proposition 1.10. For every $0 < \alpha < \alpha(\Omega)$, we have $-\varrho \leq C_\alpha \delta^\alpha$. Theorem 1.8 then yields

$$D_B(z_0, z) \geq \alpha |\log \delta(z)|$$

as $z \rightarrow \partial\Omega$. Thus it suffices to show

$$(6.1) \quad d_K(z, z_0) \leq C |\log \delta(z)|$$

as $z \rightarrow \partial\Omega$. To see this, let F_K be the Kobayashi-Royden metric. Since F_K is decreasing under holomorphic mappings, we conclude that $F_K(z; X)$ is dominated by the KR metric of the ball $B(z, \delta(z))$. Thus $F_K(z; X) \leq C|X|/\delta(z)$, from which (6.1) immediately follows (compare the proof of Proposition 7.3 in [20]). □

7. APPENDIX: EXAMPLES OF DOMAINS WITH POSITIVE HYPERCONVEXITY INDICES

In this appendix, we provide two classes of domains with $\alpha(\Omega) > 0$. The boundaries of these domains might be very irregular.

A domain $\Omega \subset \mathbb{C}^n$ is called \mathbb{C} -convex if $\Omega \cap L$ is a simply-connected domain in L for every affine complex line L . Clearly, every convex domain is \mathbb{C} -convex.

Proposition 7.1. *If $\Omega \subset \mathbb{C}^n$ is a bounded \mathbb{C} -convex domain, then $\alpha(\Omega) \geq 1/2$.*

Proof. Let $w \in \Omega$ be an arbitrarily fixed point. Let w^* be a point on $\partial\Omega$ satisfying $\delta(w) = |w - w^*|$. Let L be the complex line determined by w and w^* . Since every \mathbb{C} -convex domain is linearly convex (cf. [35], Theorem 4.6.8), it follows that there exists an affine complex hyperplane $H \subset \mathbb{C}^n \setminus \Omega$ with $w^* \in H$. Since $|w - w^*| = \delta(w)$, so H has to be *orthogonal* to L . Let π_L denote the natural projection $\mathbb{C}^n \rightarrow L$. Notice that $\pi_L(\Omega)$ is a bounded simply-connected domain in L in view of [35], Proposition 4.6.7. By Proposition 7.3 in [20], there exists a negative continuous function ρ_L on $\pi_L(\Omega)$ with

$$(\delta_L/\delta_L(z_L^0))^2 \leq -\rho_L \leq (\delta_L/\delta_L(z_L^0))^{1/2}$$

where δ_L denotes the boundary distance of $\pi_L(\Omega)$ and $z_L^0 \in \pi_L(\Omega)$ satisfies $\delta_L(z_L^0) = \sup_{\pi_L(\Omega)} \delta_L$. Fix a point $z^0 \in \Omega$. We have

$$\delta_L(z_L^0) \geq \delta_L(\pi_L(z^0)) \geq \delta(z^0).$$

Set

$$\varrho_{z_0}(z) = \sup\{u(z) : u \in PSH^-(\Omega), u(z^0) \leq -1\}.$$

Clearly, $\varrho_{z_0} \in PSH^-(\Omega)$. Since $\Omega \subset \pi_L^{-1}(\pi_L(\Omega))$, it follows that $\pi_L^*(\rho_L) \in PSH^-(\Omega)$. Since $\pi_L^*(\delta_L)(w) = \delta(w)$ and

$$\pi_L^*(\rho_L)(z^0) = \rho_L(\pi_L(z^0)) \leq -(\delta_L(\pi_L(z^0))/\delta_L(z_L^0))^2$$

so

$$\begin{aligned} \varrho_{z_0}(w) &\geq (\delta_L(z_L^0)/\delta_L(\pi_L(z^0)))^2 \pi_L^*(\rho_L)(w) \\ &\geq -(\delta_L(z_L^0)^{3/2}/\delta_L(\pi_L(z^0))^2) \delta(w)^{1/2} \\ &\geq -(R^{3/2}/\delta(z^0)^2) \delta(w)^{1/2} \end{aligned}$$

where $R = \text{diam}(\Omega)$. Thus $\alpha(\Omega) \geq 1/2$. \square

Complex dynamics also provides interesting examples of domains with $\alpha(\Omega) > 0$. Let $q(z) = \sum_{j=0}^d a_j z^j$ be a complex polynomial of degree $d \geq 2$. Let q^n denote the n -iterates of q . The attracting basin at ∞ of q is defined by

$$F_\infty := \{z \in \overline{\mathbb{C}} : q^n(z) \rightarrow \infty \text{ as } n \rightarrow \infty\},$$

which is a domain in $\overline{\mathbb{C}}$ with $q(F_\infty) = F_\infty$. The Julia set of q is defined by $J := \partial F_\infty$. It is known that J is always uniformly perfect. Thus $\alpha(F_\infty) > 0$.

We say that q is *hyperbolic* if there exist constants $C > 0$ and $\gamma > 1$ such that

$$\inf_j |(q^n)'| \geq C\gamma^n, \quad \forall n \geq 1.$$

Consider a holomorphic family $\{q_\lambda\}$ of hyperbolic polynomials of constant degree $d \geq 2$ over the unit disc Δ . Let F_∞^λ denote the attracting basin at ∞ of q_λ and let $J_\lambda := \partial F_\infty^\lambda$. Let Ω_r denote the total space of F_∞^λ over the disc $\Delta_r := \{z \in \mathbb{C} : |z| < r\}$ where $0 < r \leq 1$, that is

$$\Omega_r = \{(\lambda, w) : \lambda \in \Delta_r, w \in F_\infty^\lambda\}.$$

Proposition 7.2. *For every $0 < r < 1$, Ω_r is a bounded domain in \mathbb{C}^2 with $\alpha(\Omega_r) > 0$.*

Proof. We first show that Ω_r is a domain. Mañé, Sad and Sullivan [42] showed that there exists a family of maps $\{f_\lambda\}_{\lambda \in \Delta}$ such that

- (1) $f_\lambda : J_0 \rightarrow J_\lambda$ is a homeomorphism for each $\lambda \in \Delta$;
- (2) $f_0 = \text{id}|_{J_0}$;
- (3) $f(\lambda, z) := f_\lambda(z)$ is holomorphic on Δ for each $z \in J_0$;
- (4) $q_\lambda = f_\lambda \circ q_0 \circ f_\lambda^{-1}$ on J_λ , for each $\lambda \in \Delta$.

In other words, properties (1) \sim (3) say that $\{f_\lambda\}_{\lambda \in \Delta}$ gives a *holomorphic motion* of J_0 . By a result of Slodkowski [51], $\{f_\lambda\}_{\lambda \in \Delta}$ may be extended to a holomorphic motion $\{\tilde{f}_\lambda\}_{\lambda \in \Delta}$ of $\overline{\mathbb{C}}$ such that

- a) $\tilde{f}_\lambda : \overline{\mathbb{C}} \rightarrow \overline{\mathbb{C}}$ is a quasiconformal map of dilatation $\leq \frac{1+|\lambda|}{1-|\lambda|}$, for each $\lambda \in \Delta$;
- b) $\tilde{f}_\lambda : F_\infty^0 \rightarrow F_\infty^\lambda$ is a homeomorphism for each $\lambda \in \Delta$;
- c) $\tilde{f}(\lambda, z) := \tilde{f}_\lambda(z)$ is jointly Hölder continuous in (λ, z) .

It follows immediately that Ω_r is a domain in \mathbb{C}^n for each $r \leq 1$. Let δ_λ and δ denote the boundary distance of F_∞^λ and Ω_1 respectively. We claim that for every $0 < r < 1$ there exists $\gamma > 0$ such that

$$(7.1) \quad \delta_\lambda(w) \leq C\delta(\lambda, w)^\gamma, \quad \lambda \in \Delta_r, w \in F_\infty^\lambda.$$

To see this, choose $(\lambda', w_{\lambda'})$ where $w_{\lambda'} \in J_{\lambda'}$, such that

$$\delta(\lambda, w) = \sqrt{|\lambda - \lambda'|^2 + |w - w_{\lambda'}|^2}.$$

Write $w_{\lambda'} = \tilde{f}(\lambda', z_0)$ where $z_0 \in J_0$. Since $\tilde{f}(\lambda, z_0) \in J_\lambda$, it follows that

$$\begin{aligned} \delta_\lambda(w) &\leq |w - \tilde{f}(\lambda, z_0)| \leq |w - w_{\lambda'}| + |\tilde{f}(\lambda', z_0) - \tilde{f}(\lambda, z_0)| \\ &\leq |w - w_{\lambda'}| + C|\lambda - \lambda'|^\gamma \\ &\leq \delta(\lambda, w) + C\delta(\lambda, w)^\gamma \\ &\leq C'\delta(\lambda, w)^\gamma \end{aligned}$$

where γ is the order of Hölder continuity of \tilde{f} on Ω_r .

Recall that the Green function $g_\lambda(w) := g_{F_\infty^\lambda}(w, \infty)$ at ∞ of F_∞^λ satisfies

$$(7.2) \quad g_\lambda(w) = \lim_{n \rightarrow \infty} d^{-n} \log |q_\lambda^n(w)|, \quad w \in F_\infty^\lambda$$

where the convergence is uniform on compact subsets of F_∞^λ (cf. [48], Corollary 6.5.4). Actually the proof of Corollary 6.5.4 in [48] shows that the convergence is also uniform on compact subsets of Ω_1 . Since $\log |q_\lambda^n(w)|$ is psh in (λ, w) , so is $g(\lambda, w) := g_\lambda(w)$. By (7.1) it suffices to verify that for every $0 < r < 1$ there are positive constants C, α such that $-g_\lambda(w) \leq C\delta_\lambda(w)^\alpha$ for each $\lambda \in \Delta_r$ and $w \in F_\infty^\lambda$. This can be verified similarly as the proof of Theorem 3.2 in [15]. \square

Conjecture 7.3. *Let $D \subset \mathbb{C}$ be a domain with $\alpha(D) > 0$. Let $\{f_\lambda\}_{\lambda \in \Delta}$ be a holomorphic motion of D . Let*

$$\Omega_r := \{(\lambda, w) : \lambda \in \Delta_r, w \in f_\lambda(D)\}.$$

One has $\alpha(\Omega_r) > 0$ for each $r < 1$.

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